SNSB
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Ergodic Theory and Additive Combinatorics
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Seminar 7

(S7.1)

(i) $1_X : X \to X$, the identity on $(X, \mathcal{B}, \mu)$, is an invertible measure-preserving transformation.

(ii) The composition of two measure-preserving transformations is a measure-preserving transformation.

(iii) If $(X, \mathcal{B}, \mu, T)$ is a MPS, then $\mu(T^{-n}(A)) = \mu(A)$ for all $A \in \mathcal{B}$ and all $n \geq 1$.

(iv) If $(X, \mathcal{B}, \mu, T)$ is invertible, then $\mu(T^n(A)) = \mu(A)$ for all $A \in \mathcal{B}$ and all $n \in \mathbb{Z}$.

Proof. (i) Obviously.

(ii) Let $T : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ and $S : (Y, \mathcal{C}, \nu) \to (Z, \mathcal{D}, \eta)$ be measure-preserving and let $A \in \mathcal{D}$. Then

$$\mu((S \circ T)^{-1}(A)) = \mu(T^{-1}(S^{-1}(A))) = \nu(S^{-1}(A)) = \eta(A).$$

(iii) By induction on n.
$n = 1$ by hypothesis.
$n \Rightarrow n+1$: $\mu(T^{-n-1}(A)) = \mu(T^{-1}(T^{-n}(A))) = \mu(T^{-n}(A)) = \mu(A)$.

(iv) If $n \leq 0$ the result follows from (iii). Assume that $n \geq 1$. By induction on n.
$n = 1$: $\mu(T(A)) = \mu(T^{-1}(T(A)) = \mu(A)$.
$n \Rightarrow n+1$: $\mu(T^{n+1}(A) = \mu(T(T^n(A)) = \mu(T^n(A)) = \mu(A)$.

□

(S7.2) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be probability spaces and $T : X \to Y$ be bijective such that both T and T^{-1} are measurable. The following are equivalent

(i) T is measure-preserving.

(ii) $\mu(B) = \nu(T(B)))$ for all $B \in \mathcal{B}$.

(iii) T^{-1} is measure-preserving.

Proof. (i)⇒(ii) Assume that T is measure-preserving. Then for all $B \in \mathcal{B}$, $\mu(B) = \mu(T^{-1}(T(B))) = \nu(T(B))$.
(ii)⇔(iii) Obviously, since for all $B \in \mathcal{B}$, $(T^{-1})^{-1}(B) = T(B)$.
(ii)⇒(i) Let $C \in \mathcal{C}$. Then $T^{-1}(C) \in \mathcal{B}$ since T is measurable. Hence, $\nu(C) = \nu(T(T^{-1}(C))) = \mu(T^{-1}(C))$. □

(S7.3) Let $(X, \mathcal{B}), (Y, \mathcal{C}), (Z, \mathcal{D})$ be measurable spaces, $T : X \to Y, S : Y \to Z$ be measurable transformations.

(i) $U_{S \circ T} = U_T \circ U_S$.

(ii) U_T is linear and $U_T(f \cdot g) = (U_T f) \cdot (U_T g)$ for all $f, g \in \mathcal{M}_{\mathbb{C}}(Y, \mathcal{C})$.

(iii) If $f : Y \to \mathbb{C}, f(y) = c$ is a constant function, then $U_T(f)(x) = c$ for every $x \in X$.

(iv) $U_T(\mathcal{M}_{\mathbb{R}}(Y, \mathcal{C})) \subseteq \mathcal{M}_{\mathbb{R}}(X, \mathcal{B})$.

(v) If $f \in \mathcal{M}_{\mathbb{R}}(Y, \mathcal{C})$ is nonnegative, then $U_T f$ is nonnegative too, hence U_T is a positive operator.

(vi) For all $C \in \mathcal{C}$, $U_T(\chi_C) = \chi_{T^{-1}(C)}$.

(vii) If f is a simple function in $\mathcal{M}_{\mathbb{C}}(Y, \mathcal{C})$, $f = \displaystyle\sum_{i=1}^{n} c_i \chi_{C_i}$, $c_i \in \mathbb{C}, C_i \in \mathcal{C}$, then $U_T f$ is a simple function in $\mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$, $U_T f = \displaystyle\sum_{i=1}^{n} c_i \chi_{T^{-1}(C_i)}$.

Proof. (i) Let $f \in \mathcal{M}_{\mathbb{C}}(Z, \mathcal{D})$. Then for all $x \in X$,

$$
\begin{aligned}
U_{S \circ T} f(x) &= f((S \circ T)(x)) = f(S(Tx)) = (U_S f)(Tx)) = U_T(U_S f)(x) \\
&= (U_T \circ U_S)(f)(x).
\end{aligned}
$$

(ii) Let $f, g \in \mathcal{M}_{\mathbb{C}}(Y, \mathcal{C})$ and $\alpha, \beta \in \mathbb{C}$. For all $x \in X$, we have that

$$
\begin{aligned}
U_T(\alpha f + \beta g)(x) &= (\alpha f + \beta g)(Tx) = \alpha f(Tx) + \beta g(Tx) \\
&= \alpha U_T(f(x)) + \beta U_T(g(x)) = (\alpha U_T(f) + \beta U_T(g))(x), \\
U_T(f \cdot g)(x) &= (f \cdot g)(Tx) = \big(f(Tx)\big) \cdot \big(g(Tx)\big) = (U_T f)(x) \cdot (U_T g)(x) \\
&= \big((U_T f) \cdot (U_T g)\big)(x).
\end{aligned}
$$

(iii) Obviously.

(iv) Obviously, since $f : Y \to \mathbb{R}$ implies $f \circ T : X \to \mathbb{R}$.

(v) Obviously.

(vi) follows easily.

(vii) Apply (vi).

□

(S7.4) Let $(X, \mathcal{B})$ be a measurable space and $T : X \to X$ be measurable.

(i) $U_{1_X} = 1_{\mathcal{M}_{\mathbb{C}}(X,\mathcal{B})}$

(ii) $U_{T^n} = (U_T)^n$ for all $n \in \mathbb{N}$.

(iii) If $T : X \to X$ is bijective and both T and T^{-1} are measurable, then U_T is invertible and its inverse is $U_{T^{-1}}$. Furthermore, $U_{T^n} = (U_T)^n$ for all $n \in \mathbb{Z}$.

Proof. Easy. □

(S7.5) Let (X, d) be a compact metric space and $T : X \to X$ be a continuous mapping. For all $l \geq 1$, there exists a multiply recurrent point for $T, T^2, \ldots, T^l$.

Proof. We use a lifting trick to reduce to the case where $T : X \to X$ is a homeomorphism. Let us consider $X^{\mathbb{Z}}$ with the product topology, and the shift

$$S : X^{\mathbb{Z}} \to X^{\mathbb{Z}}, \quad (S\mathbf{x})_n = x_{n+1}.$$

It is easy to see that $(X^{\mathbb{Z}}, S)$ is an invertible TDS, and, moreover $X^{\mathbb{Z}}$ is metrizable, by B.7.6. Let

$$\tilde{X} = \{\mathbf{x} \in X^{\mathbb{Z}} \mid Tx_n = x_{n+1} \text{ for all } n \in \mathbb{Z}\}. \tag{D.1}$$

We shall prove that $\tilde{X}$ is a nonempty closed strongly S-invariant subset of $X^{\mathbb{Z}}$.

If $\mathbf{x} \in \tilde{X}$, then $T(S\mathbf{x})_n = Tx_{n+1} = x_{n+2} = T(S\mathbf{x})_{n+1}$, hence $S\mathbf{x} \in \tilde{X}$. Furthermore, if $\mathbf{y} = T^{-1}\mathbf{x}$, then $Ty_n = Tx_{n-1} = x_n = y_{n+1}$, hence $\mathbf{y} \in \tilde{X}$. Thus, $S(\tilde{X}) = \tilde{X}$, so $\tilde{X}$ is strongly S-invariant.

It is easy to see that $\tilde{X}$ is closed. If $(\mathbf{x}^{(k)})$ is a sequence in $\tilde{X}$ and $\mathbf{x} \in X^{\mathbb{Z}}$ is such that $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$, then given $n \in \mathbb{Z}$, we have that

$$Tx_n = T(\lim_{k\to\infty} \mathbf{x}_n^{(k)}) = \lim_{k\to\infty} T(\mathbf{x}_n^{(k)}) = \lim_{k\to\infty} \mathbf{x}_{n+1}^{(k)} = x_{n+1}.$$

It remains to prove that $\tilde{X}$ is nonempty. Let $x \in X$, and for each $p \geq 1$, define $\mathbf{z}^P \in X^{\mathbb{Z}}$ by

$$z_n^p = \begin{cases} T^{n+p}x & \text{if } n \geq -p \\ \text{arbitrarily} & \text{if } n < -p. \end{cases}$$

Thus, $Tz_n^p = z_{n+1}^p$ for all $n \geq -p$.

Since $X^{\mathbb{Z}}$ is a compact metrizable space, it is sequentially compact, hence $(\mathbf{z}^p)$ has a convergent subsequence. Thus, $\lim_{k\to\infty} \mathbf{z}^{p_k} = \mathbf{z}$ for some $\mathbf{z} \in X^{\mathbb{Z}}$ and some strictly increasing sequence $p_1 < p_2 < \ldots$.

We shall prove that $\mathbf{z} \in \tilde{X}$. Let $n \in \mathbb{Z}$, and $K \geq 1$ be such that $p_K \geq |n|$. Then $n \geq -p_k$, hence $Tz_n^{p_k} = z_{n+1}^{p_k}$ for all $k \geq K$. By letting $k \to \infty$, we get that $Tz_n = z_{n+1}$.

It follows that $\tilde{X}$ is a compact metric space and $S : \tilde{X} \to \tilde{X}$ is a homeomorphism. We can apply now Corollary 1.7.3 for $(\tilde{X}, S)$ to get a point $\mathbf{x} \in \tilde{X}$, and a sequence (n_k) in $\mathbb{N}$ with $\lim_{k\to\infty} n_k = \infty$ such that

$$\lim_{k\to\infty} S^{n_k}\mathbf{x} \;=\; \lim_{k\to\infty} S^{2n_k}\mathbf{x} = \ldots = \lim_{k\to\infty} S^{ln_k}\mathbf{x} = \mathbf{x}.$$

Let $p \in \mathbb{Z}$, and $i = 1, \ldots, l$. Then

$$\begin{aligned} \lim_{k\to\infty} T^{in_k} x_p &= \lim_{k\to\infty} x_{p+ink} \quad \text{since } \mathbf{x} \in \tilde{X} \\ &= \lim_{k\to\infty} \left(S^{in_k}\mathbf{x}\right)_p = x_p. \end{aligned}$$

Thus, each component of $\mathbf{x}$ is a multiply recurrent point for $T, T^2, \ldots, T^l$. □

(S7.6) For any $A \in \mathcal{B}$, let us recall that

$$\limsup_{n\to\infty} T^{-n}(A) = \bigcap_{n\geq 1} \bigcup_{i\geq n} T^{-i}(A).$$

Then

(i) $\limsup_{n\to\infty} T^{-n}(A)$ is T-invariant.

(ii) $\mu(A \Delta \limsup_{n\to\infty} T^{-n}(A)) \leq \sum_{k=1}^{\infty} k\mu(A\Delta T^{-1}(A))$. In particular, $\mu(A\Delta T^{-1}(A)) = 0$ implies $\mu(A\Delta \limsup_{n\to\infty} T^{-n}(A)) = 0$.

Proof. By A.2.7.(i) we have that

$$\begin{aligned} \limsup_{n\to\infty} T^{-n}(A) &= \{x \in X \mid x \in T^{-n}(A) \text{ for infinitely many } n\} \\ &= \{x \in X \mid T^n x \in A \text{ for infinitely many } n\}. \end{aligned}$$

Since A is measurable, we have that $T^{-n}(A)$ is measurable, hence $\limsup_{n\to\infty} T^{-n}(A)$ is measurable, by C.2.2.(iii).

(i) Let $x \in X$. Then $x \in T^{-1}\left(\limsup_{n\to\infty} T^{-n}(A)\right)$ iff $Tx \in \limsup_{n\to\infty} T^{-n}(A)$ iff $T^{n+1}x \in A$ for infinitely many n iff $x \in \limsup_{n\to\infty} T^{-n}(A)$. Thus, $\limsup_{n\to\infty} T^{-n}(A)$ is T-invariant.

(ii) Let us note that

(a) if $x \in \limsup_{n\to\infty} T^{-n}(A) \setminus A$, then there exists some k such that $x \in T^{-k}(A) \setminus A$;

(b) if $x \in A \setminus \limsup_{n\to\infty} T^{-n}(A)$, then there exists some k such that $x \notin T^{-k}(A)$, hence $x \in A \setminus T^{-k}(A)$.

Thus $A\Delta \limsup_{n\to\infty} T^{-n}(A) \subseteq \bigcup_{k\geq 1} A\Delta T^{-k}(A)$. It follows that

$$\begin{aligned}
\mu(A\Delta \limsup_{n\to\infty} T^{-n}(A)) &\leq \mu\left(\bigcup_{k\geq 1} A\Delta T^{-k}(A)\right) \leq \sum_{k=1}^{\infty} \mu(A\Delta T^{-k}(A)) \\
&\leq \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} \mu(T^{-i}(A)\Delta T^{-i-1}(A)) \\
&\quad \text{by repeatedly applying the "triangle" inequality C.4.4.(vi)} \\
&= \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} \mu(T^{-i}(A\Delta T^{-1}(A))) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{k-1} \mu(A\Delta T^{-1}(A)) = \sum_{k=1}^{\infty} k\mu(A\Delta T^{-1}(A))
\end{aligned}$$

since T is measure-preserving.

□